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Geometric superfield approach to superconformal mechanics

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Abstract. The new general geometric approach to $d=1$ conformally invariant systems, previously elaborated by us with an example of conformal mechanics, is applied in the supersymmetry case. We construct a manifestly invariant superfield description of the superconformal mechanics (SCM) models for arbitrary even N, N being a number of independent real $d=1$ Poincaré supersymmetries. These systems are shown to result from non-linear realisations of $d=1$ superconformal groups $\mathrm{SU}(1,1|N/2)$ in the cosets $\mathrm{SU}(1,1|N/2)/\mathrm{U}(N/2)$. For the $N=4$ case, which has previously been worked out only on shell, we find two different off-shell formulations related via a duality transformation. The systems with higher N are essentially new. An effect of creating the U(1) central charge in the $d=1$, $N=4$ superconformal algebra $\mathrm{su}(1,1|2)$ by the duality transformation is revealed. By extending the procedure employed in the bosonic case we derive general superfield solutions of the $N=2$ and $N=4$ SCM equations.

1. Introduction

Conformal mechanics [1] and its supersymmetric extensions [2, 3] (hereafter referred to as CM and SCM) are instructive to explore for several reasons. Being $d=1$ prototypes of conformal field theory, these systems offer an appropriate laboratory for getting some insight into the structure of conformal theories in higher dimensions. Furthermore, they provide non-trivial examples of the particle and superparticle models which have recently received much attention as the toy models for strings and superstrings. It is also worth recalling that different versions of supersymmetric quantum mechanics [4-7] describe non-trivial reductions of some four-dimensional theories of interest, such as supersymmetric Yang-Mills and supergravity theories [5, 6], and may bear a deep relation to more realistic models [7].

In the previous paper [8], we have found that the purely bosonic CM exhibits interesting geometric features. Its field equation can be interpreted as defining a class of geodesics in the group space of $d=1$ conformal group SO(1, 2). Thus, there was revealed an intimate relation between CM and the geometry of group SO(1, 2). Our consideration relied heavily upon the $d=1$ version of the covariant reduction method which was originally invented by us to deduce new superextensions of the $d=2$ Liouville and Wess-Zumino-Novikov-Witten models [9]. As has been mentioned in [8], this approach admits an immediate generalisation to the $d=1$ supersymmetry case, where it can be used to construct manifestly invariant superfield formulations of the known SCM models and to set up new models of this kind. To date, merely the $N=2$ and $N=4$ SCM† were known, the latter one only in component form.

† By N we denote the number of *real* $d=1$ Poincaré supercharges. So $N=2$ and $N=4$ in our terminology correspond to $N=1$ and $N=2$ of [2-6].

In the present paper we study implications of the $d=1$ covariant reduction techniques for the SCM models. These systems prove to be related to the geometry of appropriate coset manifolds of $d=1$ superconformal groups. In particular, the superfield equations of $N=2$ and $N=4$ SCM are shown to single out the $(1|2)$- and $(1|4)$-dimensional geodesic subspaces in the coset manifolds $\mathrm{SU}(1,1|1)/\mathrm{U}(1)$ and $\mathrm{SU}(1,1|2)/\mathrm{SU}(2)$, respectively. We present, for the first time, two different off-shell superfield formulations of $N=4$ SCM. They are related via a $d=1$ duality transformation. One of those yields the known component version of $N=4$ SCM [3], while the other gives rise to a different version. One more new result is the construction of previously unknown higher-N SCM models. They are associated with cosets $\mathrm{SU}(1,1|\mathrm{N}/2)/\mathrm{U}(\mathrm{N}/2)$ (N even). We derive the relevant equations of motion, both in the superfield and component forms, and write down the invariant physical component actions.

The paper is organised as follows. In § 2, we briefly review the $d=1$ covariant reduction method in application to the bosonic CM model ($N=0$ SCM). In § 3, the basic peculiarities of supersymmetric generalisation of our procedure are illustrated by a simple example of $N=2$ SCM. The geometric superfield formulations of $N=4$ SCM are constructed in §§ 4 and 5. Some unusual features of the $d=1$ duality transformation are discussed, among them the creation of an operator central charge in $d=1$, $N=4$ superconformal algebra. In § 6, we explain how to get the general superfield solutions of the SCM equations in the geometric formalism. Section 7 deals with higher-N SCM models. Section 8 collects concluding remarks. In particular, an interpretation of the SCM equations as integrability conditions is presented. Appendices 1 and 2 treat some technical points.

2. Preliminaries: conformal mechanics

Before considering the supersymmetry case we recall basic facts about the geometry of purely bosonic CM [1] following our paper [8].

The action and field equations of this $d=1$ model are

$$S=\lambda^{-2}\int \mathrm{d}t((\dot{\rho})^2-m^2\rho^{-2}) \qquad [\lambda^2]=\mathrm{cm}^{-1}, [m^2]=\mathrm{cm}^{-2} \tag{2.1}$$

$$\ddot{\rho}(t)=m^2\rho^{-3}. \tag{2.2}$$

They are invariant under $d=1$ conformal transformations:

$$\delta t=a+bt+ct^2\equiv f(t) \qquad \delta\rho(t)=\tfrac{1}{2}\dot{f}(t)\rho(t) \tag{2.3}$$

generators of which form the $d=1$ conformal algebra so(1, 2):

$$\mathrm{i}[L_n, L_m]=(n-m)L_{n+m} \qquad n, m=-1, 0, 1. \tag{2.4}$$

The basic observation of [8] was that equation (2.2) can be deduced in a purely geometric way, starting with a non-linear realisation of group SO(1, 2). Consider an element of SO(1, 2) parametrised as

$$g(t, z(t), u(t))\equiv \exp(\mathrm{i}tL_{-1})\exp(\mathrm{i}z(t)L_1)\exp(\mathrm{i}u(t)L_0). \tag{2.5}$$

Left action of SO(1, 2) on these elements produces for t and $\rho(t)\equiv\exp(\tfrac{1}{2}u(t))$ just transformations (2.3). Further, let us construct Cartan 1-forms:

$$g^{-1}\,\mathrm{d}g=\mathrm{i}\omega_n L_n \tag{2.6}$$

and impose on them the convariant reduction constraint:

$$\omega_n L_n = \omega_{-1}(L_{-1} + m^2 L_1) \equiv \omega_{-1} R_0 \equiv \omega_{\mathrm{R}} \in \mathrm{so}(2). \tag{2.7}$$

This condition amounts to the set of equations on the group parameters

$$\begin{aligned} z(t) &= \rho^{-1}\dot{\rho} \\ \dot{z}(t) + (z(t))^2 &= m^2\rho^{-4} \end{aligned} \tag{2.8}$$

which is easily recognised to be equivalent to (2.2).

Conditions (2.7) and (2.8) have a transparent geometric meaning. In the SO(1, 2) group manifold $\{t, z, u\}$ they single out a curve which is produced from an arbitrary fixed point of the manifold by the right action of the one-parameter subgroup SO(2) with generator R_0. Such curves are known to be geodesics [10]. Thus, (2.8) and, hence, the CM equation (2.2) define a class of geodesics in the SO(1, 2) group manifold. These geodesics are represented by the SO(1, 2) group elements of a special form

$$g_{\mathrm{R}}^{-1}\,\mathrm{d}g_{\mathrm{R}} = \mathrm{i}\omega_{\mathrm{R}} \tag{2.9a}$$

$$g_{\mathrm{R}} = g_0(c_1, c_2)\exp(\mathrm{i}\tau R_0) \tag{2.9b}$$

g_0 being a representative of the coset SO(1, 2)/SO(2) and c_1, c_2 arbitrary constant coset parameters. These constants, together with the coupling constant m, specify an initial point on the geodesic and the tangent vector at this point, while τ is the natural parameter (proper time) along the curve. Expression (2.9*b*) furnishes equation (2.2) with the general solution and so gives a purely geometric method of integrating this equation.

It is straightforward to adapt the above construction for obtaining supersymmetric extensions of equation (2.2). One has to enlarge SO(1, 2) to an appropriate $d = 1$ superconformal group, to construct a non-linear realisation of the latter and to single out, in the relevant supergroup manifold, a geodesic submanifold which properly extends geodesic (2.9).

3. $N = 2$ superconformal mechanics

As a first non-trivial example of the application of our procedure to supersymmetric $d = 1$ systems we will reproduce here, on purely geometric grounds, the superfield formulation of $N = 2$ SCM [2].

The algebra of the $d = 1$, $N = 2$ superconformal group is the Lie superalgebra $\mathrm{su}(1, 1|1) \sim \mathrm{osp}(2|2)$ [11]†

$$\mathrm{i}[L_n, L_m] = (n - m)L_{n+m} \tag{3.1a}$$

$$\{G_r, \bar{G}_q\} = -2L_{r+q} - 2(r - q)T \tag{3.1b}$$

$$\mathrm{i}[L_n, G_r] = (\tfrac{1}{2}n - r)G_{n+r} \qquad \mathrm{i}[L_n, \bar{G}_r] = (\tfrac{1}{2}n - r)\bar{G}_{n+r} \tag{3.1c}$$

$$\mathrm{i}[T, G_r] = \tfrac{1}{2}G_r \qquad \mathrm{i}[T, \bar{G}_r] = -\tfrac{1}{2}\bar{G}_r \tag{3.1d}$$

$$[T, L_n] = \{G_r, G_q\} = \{\bar{G}_r, \bar{G}_q\} = 0 \tag{3.1e}$$

$$(n, m = -1, 0, 1;\ r, q = \pm\tfrac{1}{2}).$$

† The simplest superextension of SO(1, 2) is the supergroup Osp(2|1) corresponding to $N = 1$, $d = 1$ superconformal symmetry. However, no non-trivial $N = 1$ extension of (2.2) exists.

Besides the $d=1$ conformal generators L_n, this superalgebra includes the $d=1$, $N=2$ Poincaré supersymmetry generators $G_{-1/2}$, $\bar{G}_{-1/2}$, the generators $G_{1/2}$, $\bar{G}_{1/2}$ of superconformal boosts, and the internal U(1) automorphism generator T.

In constructing non-linear realisations of SU(1, 1|1), we adopt the following two natural requirements.

(i) We wish to have a manifest $d=1$, $N=2$ supersymmetry. So the time coordinate t associated with the generator L_{-1} has to be completed to the $d=1$, $N=2$ superspace $\{t, \theta, \bar{\theta}\}$, with θ, $\bar{\theta}$ being mutually conjugated Grassmann coordinates appearing as the supergroup parameters associated with the Poincaré supersymmetry generators $G_{-1/2}$, $\bar{G}_{-1/2}$. All the other SU(1, 1|1) parameters are regarded as superfields defined on this superspace.

(ii) Hereafter, our main interest will be in the maximally invariant situations when the internal symmetry (U(1) in the present case) is realised linearly. So we are led to consider a realisation of SU(1, 1|1) in the quotient SU(1, 1|1)/U(1).

With these remarks in mind, we implement SU(1, 1|1) as the left shifts of elements of the coset SU(1, 1|1)/U(1)

$$G(t, \theta, \bar{\theta}) = \exp(\mathrm{i}tL_{-1}) \exp(\theta G_{-1/2} + \bar{\theta}\bar{G}_{-1/2}) \exp(\mathrm{i}zL_1) \exp(\xi G_{1/2} + \bar{\xi}\bar{G}_{1/2}) \exp(\mathrm{i}uL_0) \tag{3.2}$$

$$z = z(t, \theta, \bar{\theta}) \qquad u = u(t, \theta, \bar{\theta}) \qquad \xi = \xi(t, \theta, \bar{\theta}).$$

Under this choice of parametrisation, the superspace $\{t, \theta, \bar{\theta}\}$ and the dilaton superfield $u(t, \theta, \bar{\theta})$ transform with respect to the left SU(1, 1|1) shifts as

$$\begin{aligned} \delta t &= E(t, \theta, \bar{\theta}) + \tfrac{1}{2}\bar{\theta}\bar{D}E - \tfrac{1}{2}\theta DE \\ \delta\theta &= \tfrac{1}{2}\mathrm{i}\bar{D}E(t, \theta, \bar{\theta}) \qquad \delta\bar{\theta} = -\tfrac{1}{2}\mathrm{i}DE(t, \theta, \bar{\theta}) \end{aligned} \tag{3.3}$$

$$\delta u(t, \theta, \bar{\theta}) = \dot{E}(t, \theta, \bar{\theta}) \tag{3.4}$$

where $D = \partial/\partial\theta + \mathrm{i}\bar{\theta}\partial/\partial t$, $\bar{D} = -\partial/\partial\bar{\theta} - \mathrm{i}\theta\partial/\partial t$ are covariant spinor derivatives:

$$\{D, \bar{D}\} = -2\mathrm{i}\partial/\partial t \qquad D^2 = \bar{D}^2 = 0 \tag{3.5}$$

and $E(t, \theta, \bar{\theta})$ is a superfunction, collecting all the infinitesimal parameters of $d=1$, $N=2$ superconformal transformations:

$$E(t, \theta, \bar{\theta}) = f(t) - 2\mathrm{i}(\varepsilon + \beta t)\bar{\theta} - 2\mathrm{i}(\bar{\varepsilon} + \bar{\beta}t)\theta + \theta\bar{\theta}h. \tag{3.6}$$

Here $f(t)$ is already defined in (2.3), ε, β and h are, respectively, the parameters of two supersymmetries and U(1) rotations. Note that $E(t, \theta, \bar{\theta})$ defines the superconformal transformation of the supercovariant differential Δt:

$$\begin{aligned} \Delta t &= \mathrm{d}t + \mathrm{i}\theta\, \mathrm{d}\bar{\theta} - \mathrm{i}\, \mathrm{d}\theta\bar{\theta} \\ \delta\Delta t &= \dot{E}\Delta t. \end{aligned} \tag{3.7}$$

For a reason to be made clear later, we do not need to know the explicit form of transformations of the remaining coset parameters z and ξ.

To put in force the covariant reduction method, we have first to define the corresponding covariant Cartan 1-forms. This can be done by the familiar recipe of [12]:

$$G^{-1}\,\mathrm{d}G = \mathrm{i}\omega_n L_n + \mu_r G_r + \bar{\mu}_r\bar{G}_r + \nu T \equiv \mathrm{i}\Omega \in \mathrm{su}(1, 1|1) \tag{3.8}$$

$$\begin{aligned}
&\omega_{-1}=\mathrm{e}^{-u}(\mathrm{d}t-\mathrm{i}\,\mathrm{d}\theta\bar{\theta}+\mathrm{i}\theta\,\mathrm{d}\bar{\theta})=\mathrm{e}^{-u}\Delta t\\
&\omega_0=\mathrm{d}u-2z\Delta t-2\mathrm{i}\,\mathrm{d}\bar{\theta}\xi-2\mathrm{i}\,\mathrm{d}\theta\bar{\xi}\\
&\omega_1=\mathrm{e}^{u}[\mathrm{d}z-\mathrm{i}(\mathrm{d}\xi\bar{\xi}-\xi\,\mathrm{d}\bar{\xi})+2\mathrm{i}(\mathrm{d}\theta\bar{\xi}-\xi\,\mathrm{d}\bar{\theta})z+z^2\Delta t]\\
&\mu_{-1/2}=\mathrm{e}^{-u/2}[\mathrm{d}\theta-\xi\Delta t]\qquad \bar{\mu}_{-1/2}=(\mu_{-1/2})^{+}\\
&\mu_{1/2}=\mathrm{e}^{u/2}(\mathrm{d}\xi-z\,\mathrm{d}\theta-\mathrm{i}\xi\bar{\xi}\,\mathrm{d}\theta+z\xi\Delta t)\qquad \bar{\mu}_{1/2}=(\mu_{1/2})^{+}\\
&\nu=2\,\mathrm{d}\bar{\theta}\xi-2\,\mathrm{d}\theta\bar{\xi}+2\xi\bar{\xi}\Delta t.
\end{aligned}\tag{3.9}$$

These 1-forms are defined up to arbitrary gauge U(1) transformations realised as the right shifts of elements (3.2) (the parametrisation we are using corresponds to a particular fixing of this gauge freedom).

It remains to find out how to extend the constraint (2.7). In the present case the coset parameters in (3.2) are restricted to $d=1$, $N=2$ superspace $(t, \theta, \bar{\theta})$, so these define a $(1|2)$-dimensional hypersurface in $\mathrm{SU}(1,1|1)/\mathrm{U}(1)$. The corresponding geodesic submanifold should be a special case of this hypersurface parametrised by the proper time τ already defined in (2.9*b*) and appropriate Grassmann variables η, $\bar{\eta}$. The parameter τ appears as a coordinate associated with the SO(2) generator R_0, so η, $\bar{\eta}$ should be associated with the fermionic generators promoting R_0 to a graded subalgebra of $\mathrm{su}(1,1|1)$. This subalgebra is unambiguously extracted to be

$$\mathcal{H}_{\mathrm{R}}=(\Gamma=G_{-1/2}+\mathrm{i}mG_{1/2},\bar{\Gamma}=\bar{G}_{-1/2}-\mathrm{i}m\bar{G}_{1/2},R_0,T)\tag{3.10}$$

$$\begin{aligned}
&\{\Gamma,\bar{\Gamma}\}=-2R_0-4\mathrm{i}mT\qquad \{\Gamma,\Gamma\}=\{\bar{\Gamma},\bar{\Gamma}\}=0\\
&[\Gamma,R_0]=m\Gamma\qquad [\bar{\Gamma},R_0]=-m\bar{\Gamma}\\
&[\Gamma,T]=\tfrac{1}{2}\mathrm{i}\Gamma\qquad [\bar{\Gamma},T]=-\tfrac{1}{2}\mathrm{i}\bar{\Gamma}.
\end{aligned}\tag{3.11}$$

As a crucial step, we are now led to put equal to zero all the Cartan forms except for those belonging to superalgebra (3.10)

$$\begin{aligned}
&\Omega=\Omega_{\mathrm{R}}\in\mathcal{H}_{\mathrm{R}}\Rightarrow\\
&\omega_0=0\\
&\omega_1=m^2\omega_{-1}\\
&\mu_{1/2}=\mathrm{i}m\mu_{-1/2}\qquad \bar{\mu}_{1/2}=-\mathrm{i}m\bar{\mu}_{-1/2}.
\end{aligned}\tag{3.12}$$

The set (3.12) is manifestly covariant with respect to both the left $\mathrm{SU}(1,1|1)$ shifts and the right gauge U(1) shifts. Note that these constraints agree with the original Maurer-Cartan equation for the $\mathrm{su}(1,1|1)$-valued 1-form (3.8). Actually, the surviving form Ω_{R} satisfies a closed Maurer-Cartan equation on the subalgebra $\mathcal{H}_{\mathrm{R}}$.

The 1-forms (3.9) involve the differentials of Grassmann variables $\mathrm{d}\theta$, $\mathrm{d}\bar{\theta}$ together with $\mathrm{d}t$. Therefore, constraints (3.12) result in a larger number of equations for the coset parameters as compared with the bosonic case (2.7) and (2.8). Now we have

$$\begin{aligned}
&z=\tfrac{1}{2}\dot{u}\\
&\xi=\tfrac{1}{2}\mathrm{i}\bar{D}u\\
&\bar{\xi}=-\tfrac{1}{2}\mathrm{i}Du
\end{aligned}\tag{3.13}$$

$$\begin{aligned}
&[D,\bar{D}]Y=2mY^{-1}\\
&Y\equiv\mathrm{e}^{\frac{1}{2}u}\qquad \delta Y=\tfrac{1}{2}\dot{E}Y.
\end{aligned}\tag{3.14}$$

Thus, as in the CM case, all the superfield coset parameters are expressed via a single object, this time the dilaton superfield $u(t, \theta, \bar{\theta})$. As we have started from the covariant constraints (3.12), the expressions (3.13) are guaranteed to agree with the original transformation properties of ξ and z. One may, if one wishes, derive these transformations using (3.13) and the transformation laws (3.3) and (3.4).

Equation (3.14) is dynamical and it is just the $N=2$ superextension of (2.2). Its identity to the one given in [2] becomes evident after passing to real Grassmann variables $\theta^1 \equiv \frac{1}{2}(\bar{\theta}+\theta)$, $\theta^2 \equiv \frac{1}{2}\mathrm{i}(\bar{\theta}-\theta)$. In components, it amounts to the set

$$
\begin{aligned}
&F = 2m\rho^{-1} \qquad && \ddot{\rho} = (m - \psi\bar{\psi})^2\rho^{-3} \\
&\dot{\psi} = \mathrm{i}m\psi\rho^{-2} \qquad && \dot{\bar{\psi}} = -\mathrm{i}m\bar{\psi}\rho^{-2}
\end{aligned}
\tag{3.15}
$$

where we have defined

$$
\rho = Y|_{\theta=0} \qquad \psi = \mathrm{i}DY|_{\theta=0} \qquad \bar{\psi} = -\mathrm{i}\bar{D}Y|_{\theta=0} \qquad F = [D, \bar{D}]Y|_{\theta=0}. \tag{3.16}
$$

The invariant action giving rise to (3.14) and (3.15) is

$$
\begin{aligned}
S &= -\tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t\,\mathrm{d}\theta\;\mathrm{d}\bar{\theta}(DY\bar{D}Y + 2m\ln Y) \\
&= \lambda^{-2}\int \mathrm{d}t[\tfrac{1}{2}(\dot{\rho})^2 - \tfrac{1}{2}\mathrm{i}\bar{\psi}\dot{\psi} + \tfrac{1}{2}\mathrm{i}\dot{\bar{\psi}}\psi + m\psi\bar{\psi}\rho^{-2} + \tfrac{1}{8}F^2 - \tfrac{1}{2}mF\rho^{-1}].
\end{aligned}
\tag{3.17}
$$

Superconformal invariance of this action can be checked most readily in the superfield notation, using the transformation rules

$$
\begin{aligned}
&\delta D = \frac{1}{2\mathrm{i}}(D\bar{D}E)D \qquad \delta\bar{D} = \frac{1}{2\mathrm{i}}(\bar{D}DE)\bar{D} \\
&\delta(\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\bar{\theta}) = 0.
\end{aligned}
\tag{3.18}
$$

The transformation laws of the component fields follow from definition (3.16).

We postpone the discussion of the geometric meaning of equations (3.14) and (3.15) to § 6 where the general superfield solutions of $N=2$ and $N=4$ SCM will be obtained by extending the procedure employed in the bosonic case.

Before closing this section, we shall describe an equivalent formulation of $N=2$ SCM in terms of complex $N=2$ chiral superfield†. This formulation is a prototype of dual complex formulation of $N=4$ SCM that will be discussed in § 5.

The possibility to define $d=1$, $N=2$ chiral superfields in a superconformally covariant way is related to the existence of chiral $d=1$, $N=2$ superspaces closed under superconformal transformations:

$$
(t_{\mathrm{L}}, \theta), (t_{\mathrm{R}}, \bar{\theta}) \qquad t_{\mathrm{L}} = t + \mathrm{i}\theta\bar{\theta} \qquad t_{\mathrm{R}} \equiv \overline{(t_{\mathrm{L}})} = t - \mathrm{i}\theta\bar{\theta} \tag{3.19}
$$

$$
\begin{aligned}
&\delta t_{\mathrm{L}} = E(t, \theta, \bar{\theta}) + \bar{\theta}\bar{D}E = f(t_{\mathrm{L}}) - 2\mathrm{i}(\bar{\varepsilon} + \bar{\beta}t_{\mathrm{L}})\theta \\
&\delta\theta = \tfrac{1}{2}\mathrm{i}DE = \varepsilon + \beta t_{\mathrm{L}} + \tfrac{1}{2}(\dot{f} + \mathrm{i}h)\theta.
\end{aligned}
\tag{3.20}
$$

Within our scheme there is a natural place for appearance of chiral superfields as the $\mathrm{SU}(1,1|1)$ coset parameters.

Let us include the U(1) generator T in the coset, i.e. consider the situation when $\mathrm{SU}(1,1|1)$ is realised by the left shifts in its whole group manifold. Then there appears a new superfield parameter associated with the generator T:

$$
G \to G\exp(\varphi(t, \theta, \bar{\theta})T). \tag{3.21}
$$

† Description of supersymmetric mechanics via $d=1$ chiral superfields as an alternative to the real superfield description [4-6] was proposed in [13].

A net effect of this modification is the shift of the inhomogeneously transforming Cartan form ν in (3.9) by $d\varphi$

$$\tilde{\nu} = \nu + d\varphi \tag{3.22}$$

which makes $\tilde{\nu}$ entirely invariant under the left action of $SU(1,1|1)$. Owing to the latter property, we are free to add to set (3.12) one more constraint:

$$\tilde{\nu} = 2im\omega_{-1} \tag{3.23}$$

while preserving the $SU(1,1|1)$ invariance. The meaning of this constraint is that one is finally left with the Cartan forms corresponding to the generators Γ, $\bar{\Gamma}$ and $R_0 + 2imT$. As follows from (3.11), these generators constitute a closed subalgebra while the generator T may be regarded as producing external automorphisms of fermionic generators Γ, $\bar{\Gamma}$. So (3.23) does not contradict the Maurer-Cartan equations on $\mathcal{H}_R$ (3.10).

The resulting set of equations for the coset parameters is most readable when written in terms of complex superfields X, $\bar{X}$:

$$X \equiv Y \exp(\tfrac{1}{2}i\varphi) \qquad \bar{X} = Y \exp(-\tfrac{1}{2}i\varphi) \tag{3.24}$$

$$\left.\begin{matrix} D\bar{X} = 0 \\ \bar{D}X = 0 \end{matrix}\right\} \Rightarrow X = X(t_L, \theta), \bar{X} = \bar{X}(t_R, \bar{\theta}) \tag{3.25a}$$

$$\frac{\partial}{\partial t} DX = 0 \qquad \frac{\partial}{\partial t} \bar{D}\bar{X} = 0 \tag{3.25b}$$

$$2i(\dot{X}\bar{X} - \dot{\bar{X}}X) - \bar{D}\bar{X}DX = 4m. \tag{3.25c}$$

Equations (3.25) are invariant under $N = 2$ superconformal transformations acting on X, $\bar{X}$ as

$$\delta X = \tfrac{1}{2}i(\bar{D}DE)X \qquad \delta\bar{X} = \tfrac{1}{2}i(D\bar{D}E)\bar{X}.$$

The remaining coset parameters z, ξ are expressed through $u = \ln X + \ln \bar{X}$ by formulae (3.13).

Thus, in the case at hand the covariant reduction leaves us with the complex chiral coset superfields X, $\bar{X}$ subject to the free equations (3.25b) and to the additional constraint (3.25c). It should be emphasised that a chiral $d = 1$, $N = 2$ superfield carries out, off shell, a different supermultiplet as compared to the real superfield Y considered before. Though both superfields contain the same number of bosonic and fermionic degrees of freedom, they differ regarding the treatment of bosonic components. In the real case, one of the bosonic fields (F) is auxiliary while both bosonic fields of X are physical ($X|_{\theta=0} \equiv \rho(t)\exp(\tfrac{1}{2}i\varphi(t))$). Nevertheless, we will see that the superfield equations (3.25) and (3.14) yield, on shell, the same equations for the fields ρ, ψ, $\bar{\psi}$. In the real case this occurs upon elimination of the auxiliary field F by its algebraic equation of motion, whereas in the complex case the same result follows upon elimination of $\dot{\varphi}$, i.e. after a partial integration of (3.25b).

We begin with explaining the meaning of constraint (3.25c). The quantities $(\dot{X}\bar{X} - X\dot{\bar{X}})$ and $\bar{D}\bar{X}DX$ entering into (3.25c) are constants by the equations of motion (3.25b). So (3.25c) serves to identify a specific combination of these dynamical constants with the 'kinematical' constant m. Keeping this in mind, let us note that the chirality conditions (3.25a) imply

$$\bar{D}\bar{X} = 2\bar{D}Y \exp(-\tfrac{1}{2}i\varphi) \qquad DX = 2DY \exp(\tfrac{1}{2}i\varphi) \tag{3.26a}$$

$$[D, \bar{D}]Y = -Y\dot{\varphi} - 2Y^{-1}\bar{D}YDY. \tag{3.26b}$$

In virtue of (3.25c) one has

$$Y\dot{\varphi}+2Y^{-1}\bar{D}YDY=-2mY^{-1}.$$

Upon substitution of this relation into (3.26b), (3.14) is regained. Thus (3.14) and (3.25) eventually give rise to the same set of component equations.

We wish to mention that the correspondence between the real superfield formulation of $N=2$ SCM and the free theory of chiral $d=1$, $N=2$ superfield has a prototype in the bosonic case. In appendix 1 we show that the bosonic CM equation (2.2) can be regarded as describing classical configurations of a free complex $d=1$ field at a fixed value of the conserved external angular momentum, viz the U(1) charge [14]. In the supersymmetry case this U(1) is just that generated by T and the expression in the LHS of (3.25c) is the relevant conserved charge.

4. $N=4$ SCM: formulation via real superfield

As before, we begin with the structure relations of $d=1$, $N=4$ superconformal algebra su(1, 1|2) [11]†. It is a straightforward extension of $N=2$ superalgebra (3.1). The basic difference consists in that the internal symmetry group U(1) of the $N=2$ case is enlarged to SU(2) and the fermionic generators form complex SU(2) doublets $G_{-1/2a}$, $\bar{G}^a_{-1/2}$, $G_{1/2a}$, $\bar{G}^a_{1/2}$:

$$\begin{aligned}
&\{G_{ra},\bar{G}_q{}^b\}=-2\delta_a{}^bL_{r+q}+2(r-q)(\tau^i)_a{}^bT^i && [T^i,T^j]=\varepsilon^{ijk}T^k\\
&\mathrm{i}[L_n,G_{ra}]=(\tfrac{1}{2}n-r)G_{n+ra} && \mathrm{i}[L_n,\bar{G}_r{}^a]=(\tfrac{1}{2}n-r)\bar{G}^a_{n+r}\\
&\mathrm{i}[T^i,G_{ra}]=-\tfrac{1}{2}(\tau^i)_a{}^bG_{rb} && \mathrm{i}[T^i,\bar{G}_r{}^a]=\tfrac{1}{2}\bar{G}_r{}^b(\tau^i)_b{}^a\\
&(a,b=1,2;\ i,j,k=1,2,3).
\end{aligned}\tag{4.1}$$

The generators L_n constitute the algebra so(1, 2); all the other commutators and anticommutators are equal to zero.

Superalgebra (4.1) displays interesting peculiarities. First, among superalgebras su(1, 1|N/2) only su(1, 1|2) possesses SU(2) as the internal symmetry [11]. All the other members of this family necessarily involve U(N/2) as the internal symmetry group, with the U(1) factor having a non-trivial action on spinor generators. One may still modify the RHS of $\{G,\bar{G}\}$ in (4.1) by adding a U(1) generator T:

$$\{G_{ra},\bar{G}_q{}^b\}\rightarrow\{G_{ra},\bar{G}_q{}^b\}-2\mathrm{i}(r-q)\delta_a{}^bT.\tag{4.2}$$

However, consistency with the Jacobi identities requires T to commute with all the SU(1, 1|2) generators, including the spinor ones. So, T is to be regarded as a central charge generator. We will see that, in the real superfield formulation of $N=4$ SCM, this generator does not manifest itself and can be consistently put equal to zero. It becomes active upon passing to the dual formulation of $N=4$ SCM (§ 5).

One more peculiarity of superalgebra (4.1) is the presence of an outer automorphism group SU$_A$(2). Its generators $V^i=-V^{i+}$ act only on spinor generators

$$\begin{aligned}
&[V^k,G_{ra\alpha}]=\tfrac{1}{2}\mathrm{i}(\tau^k)_\alpha{}^\beta G_{ra\beta}\\
&G_{ra\alpha}\equiv(G_{ra},\varepsilon_{ab}\bar{G}_r{}^b).
\end{aligned}\tag{4.3}$$

† This is the minimal $N=4$, $d=1$ superconformal algebra. It can be extended to osp(2|4). However the latter case requires a more careful analysis (see § 7).

This freedom will be used in constructing superfield formulations of $N=4$ SCM. Note that the central charge-modified superalgebra su(1, 1|2) possesses automorphisms only with respect to the third generator of $SU_A(2)$ (4.3):

$$[V^3, G_{ra}] = \tfrac{1}{2}\mathrm{i}G_{ra} \qquad [V^a, \bar{G}_r^{\,a}] = -\tfrac{1}{2}\mathrm{i}\bar{G}_r^{\,a}. \tag{4.4}$$

After these preliminary remarks let us turn to our task, almost all procedures being as in the $N=2$ case. We follow the principles listed in the beginning of § 3 and consider a non-linear realisation of SU(1, 1|2) in the coset SU(1, 1|2)/SU(2) with the elements parametrised as†

$$G(t, \theta, \bar{\theta}, \xi, \bar{\xi}, z, u) = \exp(\mathrm{i}tL_{-1}) \times \exp(\theta G_{-1/2} + \bar{\theta}\bar{G}_{-1/2}) \exp(\mathrm{i}zL_1) \exp(\xi G_{1/2} + \bar{\xi}\bar{G}_{1/2}) \exp(\mathrm{i}uL_0). \tag{4.4}$$

Here $\{t, \theta^a, \bar{\theta}_a \equiv (\theta^a)^+\}$ are coordinates of $d=1$, $N=4$ superspace and the rest of the coset parameters are $N=4$ superfields unconstrained for the moment. The superconformal transformations induced for the coordinates $\{t, \theta, \bar{\theta}\}$ and the superfield $u(t, \theta, \bar{\theta})$ by the left SU(1, 1|2) shifts look very similar to those of the $N=2$ case:

$$\delta t = E(t, \theta, \bar{\theta}) - \tfrac{1}{2}\theta^a D_a E + \tfrac{1}{2}\bar{\theta}_a \bar{D}^a E \tag{4.6a}$$

$$\delta\theta^a = \tfrac{1}{2}\mathrm{i}\bar{D}^a E \qquad \delta\bar{\theta}_a = -\tfrac{1}{2}\mathrm{i}D_a E \tag{4.6b}$$

$$\delta u = \dot{E} \tag{4.6c}$$

$$D_a = \partial/\partial\theta^a + \mathrm{i}\bar{\theta}_a\partial/\partial t \qquad \bar{D}^a = -\partial/\partial\bar{\theta}_a - \mathrm{i}\theta^a\partial/\partial t$$
$$\{D_a, \bar{D}^b\} = -2\mathrm{i}\delta_a^{\ b}\,\partial/\partial t \tag{4.7}$$

$$E(t, \theta, \bar{\theta}) = f(t) - 2\mathrm{i}(\varepsilon(t)\bar{\theta} - \theta\bar{\varepsilon}(t)) + \tfrac{1}{2}(\theta\tau^k\bar{\theta})b^k + 2(\dot{\varepsilon}\bar{\theta} + \theta\dot{\bar{\varepsilon}})\theta\bar{\theta} + \tfrac{1}{2}(\theta\bar{\theta})^2\ddot{f}$$
$$\varepsilon^a(t) = \varepsilon^a + \beta^a t. \tag{4.8a}$$

Here $f(t)$, as before, collects the SO(1, 2) parameters, ε^a and β^a correspond to Poincaré and conformal supersymmetries and b^k to internal SU(2) transformations. Note the useful identities

$$D^2E = \bar{D}^2E = 0 \qquad [D_a, \bar{D}^a]E = 0. \tag{4.8b}$$

Further steps are to construct the Cartan 1-forms and to perform the covariant reduction to the graded subalgebra, properly extending the SO(2) generator R_0 (2.7). The computations are tedious, though straightforward. Therefore we dwell merely on several basic points.

The reduction subalgebra in the case in question is su(1|2) spanned by the generators

$$R_0 \qquad T^i \qquad \Gamma_a = G_{-1/2a} + \mathrm{i}mG_{1/2a} \qquad \bar{\Gamma}^a = \bar{G}^a_{-1/2} - \mathrm{i}m\bar{G}^a_{1/2} \tag{4.9}$$

$$\{\Gamma_a, \bar{\Gamma}^b\} = -2\delta_a^{\ b}R_0 + 4\mathrm{i}m(\tau^k)_a^{\ b}T^k \qquad \{\Gamma, \Gamma\} = 0. \tag{4.10}$$

The remaining (anti)commutators are similar to those present in (3.9). The generators Γ_a, $\bar{\Gamma}^b$ are defined up to $SU_A(2)$ rotations and, in general, are parametrised by elements of the coset $SU_A(2)/U_A(1)$

$$\tilde{\Gamma}_a = \exp(\alpha^k V^k)\Gamma_a \exp(-\alpha^k V^k)$$
$$\tilde{\bar{\Gamma}}^a = \exp(\alpha^k V^k)\bar{\Gamma}^a \exp(-\alpha^k V^k) \qquad k = 1, 2 \tag{4.11}$$

† The SU(2) indices are raised and lowered with the help of invariant skew-symmetric tensors ε_{ab}, ε^{ab}. When summing over these indices, the first index is always meant to stay in a natural position, e.g. $\theta^2 = \theta^a\theta_a$, $\theta\bar{\theta} = \theta^a\bar{\theta}_a$, $\bar{\theta}^2 = \bar{\theta}_a\bar{\theta}^a$, etc.

(a rotation with V^3 merely attaches unessential phase factors to Γ_a, $\bar{\Gamma}^b$ and is thus an automorphism of (4.10)).

We perform the reduction to the superalgebra su(1|2) with the $SU_A(2)$-rotated generators $\tilde{\Gamma}_a$, $\tilde{\bar{\Gamma}}^b$. According to the general strategy, we equate to zero all the Cartan forms, except those taking values in this subalgebra. By this procedure, the superfield coset parameters $z(t, \theta, \bar{\theta})$, $\xi^a(t, \theta, \bar{\theta})$ are expressed via $u(t, \theta, \bar{\theta})$ and there also emerge differential equations for $u(t, \theta, \bar{\theta})$. Expressions for z and ξ^a are similar to their $N=2$ prototypes (3.13) so we confine ourselves to presenting the equations for $u(t, \theta, \bar{\theta})$:

$$\begin{aligned} &(D)^2 \, \mathrm{e}^u = 4mf \\ &(\bar{D})^2 \, \mathrm{e}^u = 4m\bar{f} \end{aligned} \qquad (4.12a)$$

$$[D, \bar{D}] \, \mathrm{e}^u = 8mc \qquad (4.12b)$$

$$[D_{(a}, \bar{D}_{b)}] u = 0 \qquad (4.12c)$$

where constants $c, f, \bar{f}$ are related to the $SU_A(2)$ rotation (4.11):

$$\exp[\mathrm{i}(\alpha^1\tau^1 + \alpha^2\tau^2)] = \begin{pmatrix} c & f \\ -\bar{f} & c \end{pmatrix} \qquad c^2 + f\bar{f} = 1. \qquad (4.13)$$

The set of equations (4.12) gives the sought superfield description of $N=4$ SCM. The meaning of different equations in (4.12) is as follows.

(i) Constraints (4.12a) are kinematic off-shell irreducibility conditions. In contrast to the $N=2$ dilaton superfield $u(t, \theta, \bar{\theta})$, its $N=4$ counterpart involves, from the beginning, two irreducible off-shell representations of $d=1$, $N=4$ supersymmetry. Conditions (4.12a) are reminiscent of the $d=4$, $N=1$ tensor multiplet constraints [15] (and, in fact, at $f=0$ follow from the latter by dimensional reduction $d=4$, $N=1 \to d=1$, $N=4$). They single out from $u(t, \theta, \bar{\theta})$ a 'tensor' $d=1$, $N=4$ supermultiplet. The irreducible field content of $u(t, \theta, \bar{\theta})$ implied by (4.12a) is convenient to define as

$$\begin{aligned} &e^{u/2}|_{\theta=0} = \rho(t) \qquad \tfrac{1}{2} D_b u|_{\theta=0} = \mathrm{i}\Psi_b(t)\rho^{-1} \qquad \tfrac{1}{2}\bar{D}^a u|_{\theta=0} = -\mathrm{i}\bar{\Psi}^a(t)\rho^{-1} \\ &[D^{(a}, \bar{D}^{b)}] \, \mathrm{e}^u|_{\theta=0} = A^{(ab)}(t) \qquad [D_a, \bar{D}^a] \, \mathrm{e}^u|_{\theta=0} = C(t). \end{aligned} \qquad (4.14)$$

All the higher-dimensional components are expressed as time derivatives of the irreducible ones.

(ii) An important consequence of (4.12a) is the differential constraint

$$\frac{\partial}{\partial t}([D, \bar{D}]) \, \mathrm{e}^u = 0 \Rightarrow [D, \bar{D}] \, \mathrm{e}^u|_{\theta=0} = C = \text{constant} \qquad (4.15)$$

which is a $d=1$ prototype of the transversality condition $\partial^\mu A_\mu = 0$ typical for tensor multiplets in $d=4$ [15]. Thus the role of equation (4.12b) is to fix a constant in (4.15) in terms of original parameters m and c figuring in the definition of the covariant reduction subalgebra.

(iii) Equation (4.12c) is dynamical. It serves to eliminate the auxiliary field $A^{(ab)}(t)$ and gives rise to equations of motion for the physical fields $\rho(t)$, $\Psi^a(t)$, $\bar{\Psi}_a(t)$.

Instead of writing down the component equations we give the invariant superfield action and its component form.

Taking into account the transformation law of u (4.6c), identities (4.8b) and the transformation property of $d=1$, $N=4$ superspace integration measure†

$$\delta(\mathrm{d}t\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta) = -\dot E(\mathrm{d}t\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta) \tag{4.16}$$

the invariant action is unambiguously restored to be

$$S = -\tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t\,\mathrm{d}^4\theta(u\,\mathrm{e}^u) \tag{4.17}$$

which is easily recognised as a $d=1$ prototype of the $d=4$, $N=1$ improved tensor multiplet action. The presence of constant f in constraints (4.12a) reveals itself only after passing to components‡

$$S = \tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t\{(\dot\rho)^2 - [m^2 + 2mc\bar\Psi\Psi + mf\bar\Psi\bar\Psi + m\bar f\Psi\Psi + 4(\bar\Psi\Psi)^2]\rho^{-2}$$
$$-\mathrm{i}\bar\Psi\dot\Psi + \mathrm{i}\dot{\bar\Psi}\Psi - \tfrac{1}{2}\bar\Psi_b\Psi_a A^{(ba)}\rho^{-2} - \tfrac{1}{32}A^{(ba)}A_{(ba)}\rho^{-2}\}. \tag{4.18}$$

One may check by inspection that the component field equations following from (4.18) coincide with those implied by the superfield equation (4.12c).

In terms of physical components, the action is

$$S = \tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t\{(\dot\rho)^2 - [m^2 + 2mc\bar\Psi\Psi + mf\bar\Psi\bar\Psi + m\bar f\Psi\Psi + (\bar\Psi\Psi)^2]\rho^{-2} - \mathrm{i}\bar\Psi\dot\Psi + \mathrm{i}\dot{\bar\Psi}\Psi\}. \tag{4.19}$$

For completeness, we present the supersymmetry transformations of $\rho(t)$ and $\bar\Psi^a$ leaving this action invariant:

$$\delta\rho(t) = -\mathrm{i}(\mu\Psi - \bar\Psi\bar\mu) \qquad \delta\bar\Psi^a = (\delta\Psi_a)^+$$
$$\delta\Psi_a(t) = \frac{\mathrm{i}}{\rho}(\mu\Psi + \bar\Psi\bar\mu)\Psi_a + \dot{\bar\mu}_a\rho - \bar\mu_a\dot\rho + \frac{\mathrm{i}}{\rho}\bar\mu_a(\bar\Psi\Psi + mc) + \frac{\mathrm{i}}{\rho}mf\mu_a. \tag{4.20}$$

We close this section with several comments.

First, the final component action (4.19) does not coincide with the one corresponding to the $N=4$ SCM model proposed previously [3]. It involves only one physical boson $\rho(t)$ and therefore can be regarded as a genuine extension of the CM and $N=2$ SCM actions (2.1) and (3.17). In the next section we will see that the standard version of $N=4$ SCM with two bosonic physical fields [3] emerges upon performing the duality transformation on the above action. This version is related to the one given here, much like a complex version of $N=2$ SCM is related to its real formulation (see § 3).

Second, the version of $N=4$ SCM we are considering actually displays no dependence on the choice of $\mathrm{SU}_A(2)$ constants c, f. Indeed, it is always possible to pass to $c=1$, $f=\bar f=0$ by a proper $\mathrm{SU}_A(2)$ redefinition of θ^a, $\bar\theta_b$ in (4.12) and, respectively, of Ψ^a, $\bar\Psi_b$ in action (4.19). Then the expression within square brackets in (4.19) is reduced to

$$[m + (\bar\Psi\Psi)]^2. \tag{4.21}$$

† We use the convention $\int \mathrm{d}t\,\mathrm{d}^4\theta = \int \mathrm{d}t\frac{1}{16}D^2\bar D^2$.

‡ Note that, in the $d=4$ case, the insertion of constants $f, \bar f$ into the RHS of the improved tensor multiplet constraints is forbidden by superconformal invariance.

However, these $SU_A(2)$ constants appear to be essential while going over to the dual formulation. We will see that, to any given set of c, f, $\bar{f}$, there corresponds a dual formulation which is different off shell from the others. Note that at any choice of f, $\bar{f}$ and $c=(1-f\bar{f})^{1/2}$ the set of equations (4.12) and the action (4.19) enjoy an additional invariance under the U(1) subgroup of $SU_A(2)$ acting on $\bar{\Psi}^a(t)$, $\Psi_a(t)$ as

$$\delta\bar{\Psi}^a = \mathrm{i}\alpha\left(\bar{\Psi}^a - \frac{\bar{f}}{c}\Psi^a\right) \qquad \delta\Psi_a = -\mathrm{i}\alpha\left(\Psi_a + \frac{f}{c}\bar{\Psi}_a\right). \tag{4.22}$$

In the representation (4.21) this subgroup coincides with the one generated by V^3.

Finally, to clarify the previous remark, we present a manifestly $SU_A(2)\times SU(2)$ covariant formulation of $N=4$ SCM.

Let us pass to the $SU_A(2)$-covariant notation:

$$\theta^{a\alpha} \equiv (\theta^a, \varepsilon^{ab}\bar{\theta}_b) \qquad \overline{(\theta^{a\alpha})} = \varepsilon_{ab}\varepsilon_{\alpha\beta}\theta^{b\beta} = (\bar{\theta}_a, -\theta_a)$$

$$D_{a\alpha} \equiv (D_a, \bar{D}_a) = \partial/\partial\theta^{a\alpha} + \mathrm{i}\theta_{a\alpha}\,\partial/\partial t$$

$$\Psi_{a\alpha} \equiv (\Psi_a, \bar{\Psi}_a) \qquad \overline{(\Psi_{a\alpha})} = \varepsilon^{ab}\varepsilon^{\alpha\beta}\Psi_{b\beta} = (\bar{\Psi}^a, -\Psi^a)$$

$$J_{(\alpha\beta)} \equiv \Psi^a{}_\alpha(t)\Psi_{a\beta}(t) \qquad \lambda^{(\alpha\beta)} = \begin{pmatrix} \bar{f} & c \\ c & -f \end{pmatrix}$$

$$\bar{J}_{(\alpha\beta)} = -\varepsilon^{\alpha\rho}\varepsilon^{\beta\upsilon}J_{(\rho\upsilon)} \qquad \overline{\lambda^{(\alpha\beta)}} = -\varepsilon_{\alpha\rho}\varepsilon_{\beta\upsilon}\lambda^{(\rho\upsilon)} \qquad \lambda^{(\alpha\beta)}\lambda_{(\alpha\beta)} = -2.$$

Then (4.12) and (4.19) can be rewritten as

$$\begin{aligned} &D^{a(\alpha}D_a^{\beta)}\,\mathrm{e}^u = 4m\lambda^{(\alpha\beta)} \\ &D^{(a\alpha}D_\alpha^{b)}u = 0 \end{aligned} \tag{4.12'}$$

$$S = \tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t\left[(\dot{\rho})^2 + \frac{1}{4}\mathrm{i}\dot{\Psi}^{a\alpha}\Psi_{a\alpha} - \left(m^2 + m\lambda^{(\alpha\beta)}J_{(\alpha\beta)} - \frac{1}{6}J^{(\alpha\beta)}J_{(\alpha\beta)}\right)\rho^{-2}\right]. \tag{4.19'}$$

Looking at these formulae it becomes evident that one may always pass to $\lambda^{(\alpha\beta)} = (0,1,0)$ by a proper rotation in the $SU_A(2)$ indices α, β. Besides, for arbitrary $\lambda^{(\alpha\beta)}$ there is an invariance under SO(2) rotations in the plane orthogonal to $\lambda^{(\alpha\beta)}$. They are just given by (4.22).

5. Duality transformation and complex form of $N=4$ SCM

The superfield action (4.17) exhibits a manifest supersymmetry and gives rise to a reasonable component action. However, one cannot directly vary it with respect to the superfield u to obtain the equation of motion (4.12c) because u is subjected off shell to the constraint (4.12a). A way out is to solve (4.12a) via an appropriate unconstrained prepotential. Another option we prefer to follow here is to implement (4.12a) in the action with the help of a Lagrange multiplier superfield $\Phi(t, \theta, \bar{\theta})$:

$$S = -\tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t\,\mathrm{d}^4\theta[\mathrm{e}^u u - \Phi D^2(\mathrm{e}^u - mf\theta^2) - \bar{\Phi}\bar{D}^2(\mathrm{e}^u - m\bar{f}\bar{\theta}^2)]. \tag{5.1}$$

Varying Φ, $\bar{\Phi}$, we come back to (4.17) and (4.12a). On the other hand, u is unconstrained in the action (5.1) and one may vary it before varying Φ. As a result, one gets for u the algebraic equation

$$u = D^2\Phi + \bar{D}^2\bar{\Phi} - 1. \tag{5.2}$$

Introducing the $d=1$, $N=4$ chiral superfields:

$$\begin{aligned} &\bar{D}^2\bar{\Phi}=V \qquad \bar{D}^a V=0 \Rightarrow V \equiv V(t_{\mathrm{L}}\theta) \\ &D^2\Phi=\bar{V} \qquad D_a\bar{V}=0 \Rightarrow \bar{V}\equiv \bar{V}(t_{\mathrm{R}}\bar{\theta}) \end{aligned} \tag{5.3}$$

$$\begin{aligned} &t_{\mathrm{L}}=t+\mathrm{i}\theta\bar{\theta} \qquad t_{\mathrm{R}}=(t_{\mathrm{L}})^{+}=t-\mathrm{i}\theta\bar{\theta} \\ &\delta t_{\mathrm{L}}=E+\bar{\theta}\bar{D}E \qquad \delta t_{\mathrm{R}}=E-\theta DE \end{aligned} \tag{5.4}$$

and substituting (5.2) back into the action (5.1), we arrive at a dual representation of the $N=4$ SCM action†:

$$S=\tfrac{1}{2}\lambda^{-2}\left(\int \mathrm{d}t\,\mathrm{d}^4\theta\, Y\bar{Y}-m\bar{f}\int \mathrm{d}t_{\mathrm{L}}\,\mathrm{d}^2\theta \ln Y - mf\int \mathrm{d}t_{\mathrm{R}}\,\mathrm{d}^2\bar{\theta}\ln \bar{Y}\right) \tag{5.5}$$

where

$$Y=\mathrm{e}^{V-1/2} \qquad \bar{Y}=\mathrm{e}^{\bar{V}-1/2} \qquad \mathrm{e}^{u}=Y\bar{Y}. \tag{5.6}$$

Taking account of superconformal invariance of the $d=1$, $N=4$ chiral superspace integration measures $\delta(\mathrm{d}t_{\mathrm{L}}\,\mathrm{d}^2\theta)=\delta(\mathrm{d}t_{\mathrm{R}}\,\mathrm{d}^2\bar{\theta})=0$, the action (5.5) can be checked to be invariant under

$$\begin{aligned} &\delta Y=\dot{E}_{\mathrm{L}}(t_{\mathrm{L}},\theta)Y \qquad \delta\bar{Y}=\dot{E}_{\mathrm{R}}(t_{\mathrm{R}},\bar{\theta})\bar{Y} \\ &E_{\mathrm{L}}=\tfrac{1}{2}f(t_{\mathrm{L}})+2\mathrm{i}\theta^a\bar{\varepsilon}_a(t_{\mathrm{L}}) \qquad E_{\mathrm{R}}=(E_{\mathrm{L}})^{+} \qquad \dot{E}=\dot{E}_{\mathrm{L}}+\dot{E}_{\mathrm{R}}. \end{aligned} \tag{5.7}$$

The superfield equations of motion following from (5.5) are

$$D^2Y=4mf(\bar{Y})^{-1} \qquad \bar{D}^2\bar{Y}=4m\bar{f}(Y)^{-1}. \tag{5.8}$$

In the component notation, (5.5) becomes

$$\begin{aligned} S=\tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t(&\dot{Y}_0\dot{\bar{Y}}_0+\tfrac{1}{4}\mathrm{i}\dot{\chi}\bar{\chi}-\tfrac{1}{4}\mathrm{i}\chi\dot{\bar{\chi}}+F\bar{F}-\tfrac{1}{4}m\bar{f}Y_0^{-2}\chi\chi \\ &-\tfrac{1}{4}mf(\bar{Y}_0)^{-2}\bar{\chi}\bar{\chi}-mf(\bar{Y}_0)^{-1}\bar{F}-m\bar{f}Y_0^{-1}F) \end{aligned} \tag{5.9a}$$

or, being rewritten via physical fields,

$$\begin{aligned} S=\tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t(&\dot{Y}_0\dot{\bar{Y}}_0+\tfrac{1}{4}\mathrm{i}\dot{\chi}\bar{\chi}-\tfrac{1}{4}\mathrm{i}\chi\dot{\bar{\chi}}-m^2f\bar{f}(Y_0\bar{Y}_0)^{-1} \\ &-\tfrac{1}{4}m\bar{f}Y_0^{-2}\chi\chi-\tfrac{1}{4}mf(\bar{Y}_0)^{-2}\bar{\chi}\bar{\chi}). \end{aligned} \tag{5.9b}$$

We have defined the component fields as

$$\begin{aligned} &Y_0=Y|_{\theta=0}\equiv\rho(t)\exp(\mathrm{i}\varphi(t)) \qquad \bar{\chi}^a=\mathrm{i}\bar{D}^a\bar{Y}|_{\theta=0} \qquad \chi_a=-\mathrm{i}D_aY|_{\theta=0} \\ &F=\tfrac{1}{4}D^2Y|_{\theta=0} \qquad \bar{F}=\tfrac{1}{4}\bar{D}^2\bar{Y}|_{\theta=0}. \end{aligned} \tag{5.10}$$

The physical component action (5.9b) is invariant under the following supersymmetry

† $\int \mathrm{d}t_{\mathrm{L}}\,\mathrm{d}^2\theta\equiv\frac{1}{4}\int \mathrm{d}t_{\mathrm{L}}(D_aD^a)$, $\int \mathrm{d}t_{\mathrm{R}}\,\mathrm{d}^2\bar{\theta}\equiv\frac{1}{4}\int \mathrm{d}t_{\mathrm{R}}(\bar{D}^a\bar{D}_a)$.

transformations:

$$
\begin{aligned}
\delta\rho(t) &= \frac{1}{2\mathrm{i}}(\varepsilon^a\chi_a\,\mathrm{e}^{-\mathrm{i}\varphi} - \bar{\chi}^a\bar{\varepsilon}_a\,\mathrm{e}^{\mathrm{i}\varphi}) \\
\delta\varphi(t) &= -\frac{1}{2\rho}(\varepsilon^a\chi_a\,\mathrm{e}^{-\mathrm{i}\varphi} + \bar{\chi}^a\bar{\varepsilon}_a\,\mathrm{e}^{\mathrm{i}\varphi}) \\
\delta\bar{\chi}^a(t) &= 2\dot{\varepsilon}^a\bar{Y}_0 - 2\varepsilon^a\dot{\bar{Y}}_0 + 2\mathrm{i}\bar{\varepsilon}_a m\bar{f}Y_0^{-1} \\
\delta\chi_a(t) &= 2\dot{\bar{\varepsilon}}_a Y_0 - 2\bar{\varepsilon}_a\dot{Y}_0 + 2\mathrm{i}\varepsilon_a mf\bar{Y}_0^{-1}.
\end{aligned}
\tag{5.11}
$$

The equivalence of this version of $N=4$ SCM to that given in [3] is proved in appendix 2.

Let us explain at greater length in what sense the described formulation of $N=4$ SCM is equivalent to the real one given in § 3.

First of all, the original equations (4.12) for the superfield u are satisfied with substitution of $\mathrm{e}^u = Y\bar{Y}$. However, their status is essentially different. Equation (4.12c), which was dynamical in the real superfield formulation, is now obeyed off shell as a consequence of the chirality conditions (5.3). On the contrary, constraints (4.12a) become on-shell equations in the dual formulation. Actually, these are satisfied in virtue of the equations of motion (5.8). The same concerns the constraint (4.15) following from (4.12a). One has

$$C(t,\theta,\bar{\theta}) \equiv [D,\bar{D}]\,\mathrm{e}^u = -2D_aY\bar{D}^a\bar{Y} - 4\mathrm{i}(\dot{Y}\bar{Y} - Y\dot{\bar{Y}}) \tag{5.12a}$$

$$C(t) = 8\rho^2\dot{\varphi} + 2(\bar{\chi}^a\chi_a) \tag{5.12b}$$

and

$$\dot{C}(t) = 0 \Rightarrow C(t) = \text{constant} \tag{5.13}$$

as a consequence of the equations of motion for fields $\varphi(t)$ and $\chi_a(t)$, $\bar{\chi}^a(t)$. Thus, in the dual formulation the field $C(t)$ is expressed via the derivative of the physical field $\varphi(t)$ and it is a constant only dynamically, by virtue of the equations of motion. Upon eliminating $\dot{\varphi}(t)$ by (5.13) and identifying the constant in this equation with $8m(1-f\bar{f})^{1/2}$ one gets for $\rho(t)$ and $\Psi_a(t) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi}\chi_a(t)$, $\bar{\Psi}^a(t) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\varphi}\bar{\chi}^a(t)$ precisely the same equations as those following from action (4.19). So, the $N=4$ SCM equations in a real formulation can be regarded as an invariant subset of the complex $N=4$ SCM equations which is singled out by specialising to a fixed value of the conserved quantity $C(t)$ (5.12b).

One sees that these two formulations of $N=4$ SCM are related to each other as $N=2$ SCM is related to the theory of chiral $N=2$ superfield (§ 3) and the ordinary bosonic CM to the theory of complex $d=1$ field (appendix 1). To understand the meaning of the conserved quantity $C(t)$, let us inspect in more detail the invariance properties of actions (5.5) and (5.9). The off-shell $\mathrm{U}_A(1)$ invariance (4.22) of the real $N=4$ SCM action is not respected in general by (5.5) and (5.9) (though it is restored on shell at any given fixed value of $C(t)$). Instead, these actions respect a new Abelian off-shell symmetry:

$$Y' = \mathrm{e}^{\mathrm{i}\alpha}Y \qquad \bar{Y}' = \mathrm{e}^{-\mathrm{i}\alpha}\bar{Y}. \tag{5.14}$$

This new invariance is of the same nature as, e.g., the one associated with the duality transformations in $d=4$ SUSY [15]. An interesting peculiarity of the $d=1$ case is that this symmetry proves to be naturally incorporated into the underlying superconformal symmetry. It emerges in the Lie bracket of Poincaré and conformal supersymmetry transformations of fields $\varphi(t)$ and χ_a, $\bar{\chi}^a$. As follows from (5.11):

$$\begin{aligned}(\delta_\varepsilon\delta_\beta-\delta_\beta\delta_\varepsilon)\varphi(t)&=(\varepsilon^a\bar{\beta}_a+\beta^a\bar{\varepsilon}_a)+\ldots\\(\delta_\varepsilon\delta_\beta-\delta_\beta\delta_\varepsilon)\bar{\chi}^a(t)&=-\mathrm{i}(\varepsilon^a\bar{\beta}_\gamma+\beta^a\bar{\varepsilon}_\gamma)\bar{\chi}^\gamma+\ldots.\end{aligned}\tag{5.15}$$

Comparing it with formula (4.2), we conclude that in the complex formulation of $N=4$ SCM the $N=4$ superconformal algebra is necessarily modified by an operator central charge T possessing a non-trivial action on the physical fields. The quantity $C(t)$ in (5.12) is just proportional to the conserved 'current' generating this T symmetry. The fields $\rho(t)$, $\Psi_a=\frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\varphi}\chi_a$, $\bar{\Psi}^a$ entering into the equations of the real formulation of $N=4$ SCM are inert under T. This explains why the central charge does not manifest itself in the real formulation.

One more curious feature of the $d=1$ duality transformation is related to an $\mathrm{SU_A}(2)$ freedom in the definition of constants c, f. Instead of starting with (4.12a), one might choose as the basic constraint some $\mathrm{SU_A}(2)/\mathrm{U}(1)$ mixture of (4.12a) and (4.12b):

$$\begin{aligned}&\tilde{D}\tilde{D}\,\mathrm{e}^u=4m\tilde{f}\qquad \tilde{\bar{D}}\tilde{\bar{D}}\,\mathrm{e}^u=4m\tilde{\bar{f}}\\&\tilde{\theta}^a=\cos(\alpha)\theta^a+\sin(\alpha)\,\mathrm{e}^{\mathrm{i}\gamma}\bar{\theta}^a\\&\tilde{\bar{\theta}}_a=\cos(\alpha)\bar{\theta}_a-\sin(\alpha)\,\mathrm{e}^{-\mathrm{i}\gamma}\theta_a\\&\tilde{f}=\cos^2(\alpha)f-\sin^2(\alpha)\,\mathrm{e}^{-2\mathrm{i}\gamma}\bar{f}-\sin(2\alpha)\,\mathrm{e}^{-\mathrm{i}\gamma}c.\end{aligned}\tag{5.16}$$

Inserting (5.16) into action (4.17) one arrives at a different dual action, where $\tilde{f}$ stands for f and the notion of chirality is defined with respect to $\tilde{\theta}^a$, $\tilde{\bar{\theta}}_a$:

$$u(t,\theta,\bar{\theta})=V(\tilde{t}_{\mathrm{L}}\tilde{\theta})+\bar{V}(\tilde{t}_{\mathrm{R}}\tilde{\bar{\theta}})-1.\tag{5.17}$$

Thus there exists a whole $\mathrm{SU_A}(2)/\mathrm{U_A}(1)$ orbit of dual formulations of the same real $N=4$ SCM (4.12). All those are non-equivalent off shell and correspond to different patterns of the U(1) central charge modification of $N=4$ superconformal algebra (4.1). For instance, the choice (5.16) amounts to (we ignore the SU(2) indices)

$$\begin{aligned}\{G,\bar{G}\}&\to\{G,\bar{G}\}-2\mathrm{i}(r-q)\cos(2\alpha)T\\\{G,G\}&\to\{G,G\}-2\mathrm{i}(r-q)\sin(2\alpha)\,\mathrm{e}^{\mathrm{i}\gamma}T.\end{aligned}\tag{5.18}$$

Note that the option $\tilde{f}=\tilde{\bar{f}}=0$, $\tilde{c}=1$ gives rise to the dual formulation in terms of a free chiral $N=4$ superfield.

We wish to mention that the superfield equations of $N=4$ SCM in dual formulation, including the chirality conditions, can be unambiguously deduced by applying the covariant reduction procedure to the central-charge-modified $N=4$ superconformal algebra. The consideration goes along the same lines as in the $N=2$ case (§ 3). One should put the central charge generator into the coset and perform the covariant reduction to subalgebra (4.9) enlarged by this generator.

6. Superfield form of general solution

As has already been mentioned, the covariant reduction techniques provide us with a geometric way of getting general solutions of field equations of CM and SCM. The procedure of integrating these equations is reduced to purely algebraic manipulations which are based mainly on the structure relations of relevant $d=1$ superconformal algebras.

The strategy we will keep to is a straightforward generalisation of the one employed in the bosonic CM [8], so we will not enter into details of presentation.

We begin once again with the simple case of $N=2$ SCM. The basic covariant reduction constraint is (see (3.12))

$$G_R^{-1}\,\mathrm{d}G_R = \mathrm{i}\Omega_R \in \mathscr{H}_R = \{\Gamma, \bar{\Gamma}, R_0, T\} \tag{6.1}$$

where the subalgebra $\mathscr{H}_R = \{R_0, \Gamma, \bar{\Gamma}, T\} \subset \mathrm{su}(1,1|1)$ is defined in (3.10). The most general solution of (6.1) can be written as (cf (2.9b))

$$G_R = \tilde{G}_0(c_1, c_2, \mu, \bar{\mu})\exp(\mathrm{i}\tau R_0)\exp(\eta\Gamma + \bar{\eta}\bar{\Gamma})\exp(hT) \tag{6.2}$$

where c_1, c_2, μ, $\bar{\mu}$ are constants, respectively bosonic and fermionic, and τ, η, $\bar{\eta}$, h are superfunctions given on the $d=1$, $N=2$ superspace $\{t, \theta, \bar{\theta}\}$.

The meaning of different factors in (6.2) is as follows. The element $\tilde{G}_0$ belongs to the coset $\mathrm{SU}(1,1|1)/\mathrm{H}_R$. It can be parametrised, without loss of generality, as

$$\tilde{G}_0(c_1, c_2, \mu, \bar{\mu}) = \exp(\mathrm{i}c_1 L_{-1})\exp(\mu G_{-1/2} + \bar{\mu}\bar{G}_{-1/2})\exp(\mathrm{i}c_2 L_0) \tag{6.3}$$

(any other parametrisation is related to (6.3) by a redefinition of parameters τ, η, $\bar{\eta}$, h). The factors to the right of $\tilde{G}_0$ represent the coset $\mathrm{H}_R/\mathrm{U}(1)$. The parameters $\tau(t,\theta,\bar{\theta})$, $\eta(t,\theta,\bar{\theta})$, $\bar{\eta}$ can be regarded as coordinates of a $(1|2)$-dimensional geodesic hypersurface which is embedded into the group space of $\mathrm{SU}(1,1|1)$ and extends the one-dimensional geodesic subspace (the geodesic curve) of the bosonic case. The position of this hypersurface within the $\mathrm{SU}(1,1|1)$ manifold is specified by constants c_1, c_2, μ, $\bar{\mu}$.

Since the $N=2$ SCM equation of motion (3.14) is a consequence of (6.1), the general solution of the latter immediately yields the general solution of (3.14). Comparing (6.2) with the original $\mathrm{SU}(1,1|1)/\mathrm{U}(1)$ coset element (3.2), one finds

$$\begin{aligned} t &= c_1 + \exp(c_2)\frac{1}{m}\tan m\tau - \frac{\mathrm{i}}{\cos m\tau}(\mu\bar{\eta} - \eta\bar{\mu})\exp(\tfrac{1}{2}c_2) \\ \theta &= \mu + \exp(\tfrac{1}{2}c_2)\frac{1}{\cos m\tau}\eta \qquad \bar{\theta} = \bar{\mu} + \exp(\tfrac{1}{2}c_2)\frac{1}{\cos m\tau}\bar{\eta} \end{aligned} \tag{6.4}$$

$$u = c_2 - 2\ln(\cos m\tau) + 2\eta\bar{\eta}m \tag{6.5}$$

(h is also unambiguously fixed). After expressing τ and η, $\bar{\eta}$ in terms of $\{t, \theta, \bar{\theta}\}$, one eventually gets the general solution for $u(t,\theta,\bar{\theta})$ in the form

$$\begin{aligned} \exp(u) &= aa^{+}\left(1 - \mathrm{i}\frac{b}{a}\tilde{t}_L\right)\left(1 + \mathrm{i}\frac{b}{a^{+}}\tilde{t}_R\right) \\ \tilde{t}_L &= t + \mathrm{i}\theta\bar{\theta} - 2\mathrm{i}\theta\bar{\mu} \qquad t_R = (t_L)^{+} \\ b(a + a^{+}) &- 2b^2\mu\bar{\mu} = 2m \\ a &= \exp(\tfrac{1}{2}c_2) + \mathrm{i}m(c_1 - \mathrm{i}\mu\bar{\mu})\exp(-\tfrac{1}{2}c_2) \qquad b = m\exp(-\tfrac{1}{2}c_2). \end{aligned} \tag{6.6}$$

The fact that e^u is factorised into a product of chiral and antichiral $d=1$, $N=2$ superfunctions reflects the correspondence between the equations of $N=2$ SCM and those describing a chiral $d=1$, $N=2$ superfield (see discussion in § 3).

Let us briefly discuss the transformation properties of solution (6.6) under the $N=2$ superconformal group (3.3) and (3.4). It is easy to check that the infinitesimal transformations of u at fixed t, θ, $\bar{\theta}$:

$$\delta^* u = \dot{E} - E\dot{u} - \tfrac{1}{2}\mathrm{i}\bar{D}EDu - \tfrac{1}{2}\mathrm{i}DE\bar{D}u$$

are reduced to appropriate variations of the integration constants in (6.6). For instance, under supersymmetry

$$\delta a = \mathrm{i}(\mu\bar{\beta}+\beta\bar{\mu})a+2\mu\bar{\varepsilon}b \qquad \delta b = \mathrm{i}(\beta\bar{\mu}-\mu\bar{\beta})b$$
$$\delta\mu = \varepsilon+\mathrm{i}(a/b-\mu\bar{\mu})\beta. \tag{6.7}$$

It is a simple exercise to indicate the SU(1, 1|1) generators leaving the above solution invaraint:

$$\{\hat{R}_0, \hat{\Gamma}, \hat{\bar{\Gamma}}, \hat{U}\} = \tilde{G}_0\{R_0, \Gamma, \bar{\Gamma}, U\}\tilde{G}_0^{-1} \equiv \hat{\mathscr{H}}_{\mathrm{R}}$$
$$\delta^*_{\hat{\mathscr{H}}_{\mathrm{R}}} u = 0. \tag{6.8}$$

Like in the bosonic case [8], the geometric interpretation of this invariance is that generators (6.8) produce the motions along the directions belonging to the hypersurface $\{\tau, \eta, \bar{\eta}\}$, without affecting the constants c_1, c_2, μ, $\bar{\mu}$ and, hence, preserve the shape of the hypersurface and its orientation in the SU(1, 1|1) group space. Any other SU(1, 1|1) transformations change the above constants. One may say that SU(1, 1|1) is spontaneously broken on solution (6.6) down to subgroup $\hat{\mathrm{H}}_{\mathrm{R}}$ generated by (6.8).

The $N=4$ case can be treated quite analogously. It is convenient from the beginning to fix the $\mathrm{SU}_A(2)$ freedom so as to have $f=\bar{f}=0$, $c=1$. Then the covariant reduction constraint is

$$G_{\mathrm{R}}^{-1}\,\mathrm{d}G_{\mathrm{R}} = \mathrm{i}\Omega_{\mathrm{R}} \in \mathscr{H}_{\mathrm{R}} = \{R_0, \Gamma^a, \bar{\Gamma}_a, T^i\} \tag{6.9}$$

and its general solution is given by

$$G_{\mathrm{R}} = \tilde{G}_0(c_1, c_2, \mu^a, \bar{\mu}_a)g(\tau, \eta^a, \bar{\eta}_a) \tag{6.10}$$

where $\tilde{G}_0$ and g represent, respectively, the cosets $\mathrm{SU}(1,1|2)/\mathrm{H}_{\mathrm{R}}$ and $\mathrm{H}_{\mathrm{R}}/\mathrm{SU}(2)$. The explicit form of these elements is an immediate extension of (6.2) and (6.3), so we do not present it here. The general solution looks much like (6.6):

$$\exp(u(t,\theta,\bar{\theta})) = aa^+\left(1-\mathrm{i}\frac{b}{a}\tilde{t}_{\mathrm{L}}\right)\left(1+\mathrm{i}\frac{b}{a^+}\tilde{t}_{\mathrm{R}}\right) \tag{6.11}$$

$$\tilde{t}_{\mathrm{L}} = t+\mathrm{i}\theta\bar{\theta}-2\mathrm{i}\theta\bar{\mu} \qquad \tilde{t}_{\mathrm{R}} = (t_{\mathrm{L}})^+$$
$$b(a+a^+)-2b^2\mu\bar{\mu} = 2m$$
$$a = \exp(\tfrac{1}{2}c_2)+\mathrm{i}m(c_1-\mathrm{i}\mu\bar{\mu})\exp(-\tfrac{1}{2}c_2) \qquad b = m\exp(-\tfrac{1}{2}c_2). \tag{6.12}$$

The stability subgroup of solution (6.11) is $\hat{\mathrm{H}}_{\mathrm{R}}$ related to H_{R} by means of the $\mathrm{SU}(1,1|2)/\mathrm{H}_{\mathrm{R}}$ rotation with $\tilde{G}_0(c_1, c_2, \mu, \bar{\mu})$.

7. Towards higher N

We have shown that the $N=2$ and $N=4$ SCM equations can be algorithmically deduced starting solely from the structure relations of $d=1$ superconformal algebras su(1, 1|1) and su(1, 1|2). One may wonder what happens while treating, along the same lines, the superalgebras incorporating higher-N $d=1$ supersymmetries. Here we apply our techniques to superalgebras su(1, 1|$N/2$) with arbitrary even N. The arising systems directly generalise the real $N=4$ SCM considered in § 4 and can thus be regarded as higher-N SCM models.

The (anti)commutation relations of su$(1,1|N/2)$ are [11]

$$\begin{aligned}
&\mathrm{i}[L_n, L_m] = (n-m)L_{n+m} && n, m = -1, 0, 1;\ r, q = -\tfrac{1}{2}, \tfrac{1}{2} \\
&\mathrm{i}[L_n, G_{ra}] = (\tfrac{1}{2}n - r)G_{r+n\,a} && \mathrm{i}[L_n, \bar{G}_r{}^a] = (\tfrac{1}{2}n - r)\bar{G}^a_{r+n} \\
&[T^i, G_{ra}] = \tfrac{1}{2}(\lambda^i)_a{}^b G_{rb} && [T^i, \bar{G}_r{}^a] = -\tfrac{1}{2}\bar{G}_r{}^b(\lambda^i)_b{}^a \\
&[T, G_{ra}] = \tfrac{1}{2}G_{ra} && [T, \bar{G}_r{}^a] = -\tfrac{1}{2}\bar{G}_r{}^a \\
&\{G_{ra}, \bar{G}_q{}^b\} = -2\delta_a{}^b L_{r+q} + 2(r-q)\mathrm{i}\left[(\lambda^i)_a{}^b T^i - \frac{(N-4)}{N}\delta_a{}^b T\right]
\end{aligned} \tag{7.1}$$

where $(\lambda^i)_a{}^b$ are generators of the fundamental representation of SU$(N/2)$:

$$(\lambda^i)_a{}^b(\lambda^i)_c{}^d = 2\delta_a{}^d\delta_c{}^b - (4/N)\delta_a{}^b\delta_c{}^d.$$

We see that superalgebra (7.1) at any N, except $N=4$, necessarily contains a U(1) generator T having a non-trivial action on spinor generators.

As before, we realise SU$(1,1|N/2)$ by the left shifts in the coset SU$(1,1|N/2)/$SU$(N/2)\times$U(1) and identify the coset parameters corresponding to the $d=1$ Poincaré supersymmetry generators L_{-1}, $G_{-1/2a}$, $\bar{G}^a_{-1/2}$ with the $d=1$, N superspace coordinates $\{t, \theta^a, \bar{\theta}_a\}$. We choose su$(1|N/2) = \{R_0, G_{-1/2a} + \mathrm{i}mG_{1/2a}, \bar{G}^a_{-1/2} - \mathrm{i}m\bar{G}^a_{1/2}, T, T^i\}$ as the covariant reduction subalgebra. Without entering into details of computation, let us write down the final equations for the basic superfield $u(t, \theta, \bar{\theta})$:

$$D_aD_b\,\mathrm{e}^u = 0 \qquad \bar{D}^a\bar{D}^b\,\mathrm{e}^u = 0 \tag{7.2a}$$

$$[D_a, \bar{D}^b]\,\mathrm{e}^u - 2\,\mathrm{e}^{-u}D_a\,\mathrm{e}^u\bar{D}^b\,\mathrm{e}^u + \mathrm{e}^{-u}D_c\,\mathrm{e}^u\bar{D}^c\,\mathrm{e}^u\delta_a{}^b = 4m\delta_a{}^b \tag{7.2b}$$

$$D_a = \partial/\partial\theta^a + \mathrm{i}\bar{\theta}_a\partial/\partial t \qquad \bar{D}^a = -\partial/\partial\bar{\theta}_a - \mathrm{i}\theta^a\partial/\partial t.$$

These equations are an obvious generalisation of (4.12) and reduce to the $f=\bar{f}=0$ version of the latter at $N=4$. Note that non-zero constants f, $\bar{f}$ are not allowed at $N>4$ since a non-trivial external automorphism group exists only in the special case of $N=4$. The set (7.2) is invariant under superconformal transformations which have the same form as in (3.3) and (4.6):

$$\begin{aligned}
&\delta t = E(t, \theta, \bar{\theta}) - \tfrac{1}{2}\bar{D}^aE\bar{\theta}_a - \tfrac{1}{2}\theta^aD_aE \\
&\delta\theta^a = \tfrac{1}{2}\mathrm{i}\bar{D}^aE \qquad \delta\bar{\theta}_a = -\tfrac{1}{2}\mathrm{i}D_aE \qquad \delta\,\mathrm{e}^u = \dot{E}\,\mathrm{e}^u
\end{aligned} \tag{7.3}$$

$$E(t, \theta, \bar{\theta}) = f(t) - 2\mathrm{i}(\varepsilon(t)\bar{\theta} - \theta\bar{\varepsilon}(t)) + 2(\dot{\varepsilon}\bar{\theta} + \theta\dot{\bar{\varepsilon}})\theta\bar{\theta} + \tfrac{1}{2}(\theta\bar{\theta})^2\ddot{f} + b^i(\theta\lambda^i\bar{\theta}) - \alpha\theta\bar{\theta}$$

with $f(t) = a + bt + ct^2$, $\varepsilon(t) = \varepsilon + \beta t$, b^i, α being infinitesimal parameters of SU$(1,1|N/2)$.

An essential difference from the $N=4$ case consists in that constraints (7.2a) not only reduce the off-shell component content of $u(t, \theta, \bar{\theta})$ but also partly put the system on shell. One may check that, for any even N, the θ decomposition of the superfield e^u subject to (7.2a) is as follows:

$$\mathrm{e}^u = \rho(\rho + 2\mathrm{i}\theta\Psi + 2\mathrm{i}\bar{\theta}\bar{\Psi}) + \theta^a\bar{\theta}_bc_a{}^b + 2[\theta(\rho\dot{\Psi}) + (\rho\dot{\bar{\Psi}})\bar{\theta}]\theta\bar{\theta} + \tfrac{1}{2}(\theta\bar{\theta})^2(\ddot{\rho^2}). \tag{7.4}$$

However, for $N>4$, (7.2a) imply in addition the differential constraints

$$(\dot{c}_a{}^b) = 0 \tag{7.5a}$$

$$(\rho\ddot{\Psi}) = (\rho\ddot{\bar{\Psi}}) = 0 \tag{7.5b}$$

$$(\dddot{\rho^2}) = 0 \tag{7.5c}$$

(recall that, in the $N=4$ case, an analogous constraint appeared only for the singlet piece of $c_a{}^b$ (4.15)). Fortunately, these constraints prove to be a consequence of the dynamical equations embodied in (7.2*b*):

$$\begin{aligned} &c_a{}^b + 8\bar{\Psi}^b\Psi_a - 4\delta_a{}^b\bar{\Psi}\Psi = 4m\delta_a{}^b \\ &\dot{\bar{\Psi}}^a = -\mathrm{i}\rho^{-2}\bar{\Psi}^a(m+\bar{\Psi}\Psi) \\ &\dot{\Psi}_a = \mathrm{i}\rho^{-2}\Psi_a(m+\bar{\Psi}\Psi) \\ &\ddot{\rho}(t) = \rho^{-3}(m+\bar{\Psi}\Psi)^2. \end{aligned} \tag{7.6}$$

The equations for physical fields ρ, $\bar{\Psi}^a$, Ψ_a follow from the action which is a straightforward extension of the component $N=4$ action (4.9):

$$S = \tfrac{1}{2}\lambda^{-2}\int \mathrm{d}t[(\dot{\rho})^2 - \rho^{-2}(m+\bar{\Psi}\Psi)^2 + \mathrm{i}\dot{\bar{\Psi}}\Psi - \mathrm{i}\bar{\Psi}\dot{\Psi}] \tag{7.7}$$

and is invariant under the following supersymmetry transformations:

$$\begin{aligned} &\delta\rho = -\mathrm{i}(\varepsilon^a(t)\Psi_a - \bar{\Psi}^a\bar{\varepsilon}_a(t)) \\ &\delta\bar{\Psi}^a = -\mathrm{i}\rho^{-1}[\varepsilon(t)\Psi + \bar{\Psi}\bar{\varepsilon}(t)]\bar{\Psi}^a + \dot{\varepsilon}^a\rho - \varepsilon^a\dot{\rho} - \mathrm{i}\rho^{-1}\varepsilon^a(\bar{\Psi}\Psi + m) \\ &\delta\Psi_a = \mathrm{i}\rho^{-1}[\varepsilon(t)\Psi + \bar{\Psi}\bar{\varepsilon}(t)]\Psi_a + \dot{\bar{\varepsilon}}_a\rho - \bar{\varepsilon}_a\dot{\rho} + \mathrm{i}\rho^{-1}\bar{\varepsilon}_a(\bar{\Psi}\Psi + m) \end{aligned} \tag{7.8}$$

which close on shell. Of course, it remains to learn how to divide (7.2) into the kinematical constraints and dynamical equations and how to extend the action (7.7) off shell. It would be of interest also to check whether the system (7.7) is contained in the class of $d=1$ models with N extended supersymmetry proposed in [16].

Finally, we would like to mention that the lower-N $d=1$ superconformal algebras might be extended to higher N via superalgebras osp$(2|N)$ with the bosonic part so$(1,2)\oplus$so(N) where N may be both even and odd (recall the isomorphism su$(1,1|1)\sim$osp$(2|2)$). However, we have checked that these superalgebras, beginning with $N=3$, contain no graded subalgebras which would include the generator R_0 in parallel with the SO(N) generators. Therefore, within this framework, it appears impossible to achieve non-trivial $d=1$ systems with linearly realised SO(N) symmetry. The options when only a subgroup of SO(N) corresponds to linear symmetries require a special analysis.

8. Concluding remarks

The main goal of this somewhat lengthy paper was to demonstrate the efficiency of the covariant reduction method for constructing $d=1$ superconformal models and analysing their invariance properties. We have presented a common geometric view on these models, given manifestly invariant superfield formulations of $N=4$ SCM, and deduced a new series of SCM models for arbitrary N. It remains to establish a link with models of current interest, such as superstrings, supermembranes, etc. In this connection, we would like to notice that the considered systems are similar, in some aspects, to the spinning superparticle models [17]. Indeed, their basic objects are $d=1$ superfields taking values in graded manifolds, i.e. supermanifolds. A difference is that, in the case at hand, the world-line and target superspaces are unified within a single graded manifold, the quotient SU$(1,1|N/2)/$U$(N/2)$. This analogy suggests that the models in question can likely be reproduced as fixed gauges of appropriate spinning superparticle models.

One more remark concerns an analogy with the $d=2$ super-Liouville models [9]. The superfield equations of the latter are integrable in the sense that they amount to zero-curvature representations on certain superalgebras. Our consideration shows that the superfield SCM equations do equally admit a similar interpretation.

Indeed, let us apply once again to the $N=2$ case. The basic constraint (3.12) leading to (3.14) can be equivalently replaced by the condition that the curvature of the $\mathscr{H}_R$-valued 1-superform Ω_R vanishes:

$$d_1\Omega_R(d_2)-d_2\Omega_R(d_1)+i[\Omega_R(d_1),\Omega_R(d_2)]=0 \tag{8.1}$$

where the superfield $Y(t,\theta,\bar\theta)$ in Ω_R is not subjected to (3.14) before imposing (8.1) (ξ and z are assumed to be expressed via Y by (3.13)). Decomposing Ω_R in differentials $d\theta$, $d\bar\theta$, Δt and introducing the lengthened covariant derivatives

$$\begin{aligned}&\Omega_R=d\theta\Omega_\theta-d\bar\theta\bar\Omega_\theta+\Omega_t\Delta t\\&\nabla_\theta=D+i\Omega_\theta\qquad\bar\nabla_\theta=\bar D+i\bar\Omega_\theta\qquad\nabla_t=\partial_t+i\Omega_t\end{aligned} \tag{8.2}$$

one rewrites (8.1) as the set of equations

$$\{\nabla_\theta,\nabla_\theta\}=\{\bar\nabla_\theta,\bar\nabla_\theta\}=0 \tag{8.3a}$$

$$\{\nabla_\theta,\bar\nabla_\theta\}=-2i\nabla_t \tag{8.3b}$$

$$[\nabla_\theta,\nabla_t]=0. \tag{8.3c}$$

Note that (8.3c) follows from (8.3a, b) by Bianchi identities.

So the $N=2$ SCM equation (3.14) is equivalent to the integrability conditions (8.3a, b).

The equations of higher-N SCM can be given an analogous interpretation.

Finally, an urgent problem for a future study is to carry out the quantisation of superfield SCM models and to find out how their remarkable geometric properties reveal themselves in the quantum region. Note that the component $N=4$ SCM was quantised in [3] by using its complex version. It would be of interest to see whether the dual equivalence of complex and real formulations of $N=4$ SCM survives quantisation.

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Appendix 1. Conformal mechanics and complex $d=1$ field theory

Let us show that the CM equation (2.1) can be viewed as a result of partial solving of the free equations for a $d=1$ complex field. This is a particular case of the phenomenon indicated in [14].

We start with the action

$$S=\tfrac{1}{2}\lambda^{-2}\int dt\,\dot z\dot{\bar z}=\tfrac{1}{2}\lambda^{-2}\int dt[(\dot\rho)^2+\rho^2(\dot\varphi)^2] \tag{A1.1}$$

where $z = \exp(i\varphi(t))\rho(t)$. The equations of motion are

$$\ddot{\rho}(t) = \rho(\dot{\varphi})^2 \tag{A1.2a}$$

$$(\rho^2\dot{\varphi})^{\cdot} = 0 \Rightarrow \rho^2\dot{\varphi} = \text{constant} \equiv m. \tag{A1.2b}$$

Equation (A1.2b) is the conservation law for the Noether charge $\rho^2\dot{\varphi}$ (external 'angular momentum') corresponding to U(1) transformations $z' = e^{i\alpha}z$. Choosing a definite value of m for $\rho^2\dot{\varphi}$ and expressing $\dot{\varphi}$ by (A1.2b) one gets for $\rho(t)$ just equation (2.2). Thus one concludes that (2.2) describes classical configurations of the free complex $d = 1$ field $z(t)$ at a fixed external angular momentum. Note that one might add to (A1.2) an U(1)-invariant potential term:

$$(\text{A1.1}) \to (\text{A1.1}) - \tfrac{1}{2}\lambda^{-2}\int dt(z\bar{z})^{-1}a^2. \tag{A1.3}$$

For $\rho(t)$ one would again get (2.2) but with m^2 shifted by a constant a^2. So (2.2) can equally be embedded into the theory of a self-interacting $d = 1$ complex field. This consideration clarifies the relationship between real and complex formulations of $N = 2$ and $N = 4$ SCM (§§ 3 and 5).

It is noteworthy that the dual correspondence between real and complex forms of $N = 4$ SCM has a prototype in the purely bosonic case. Let us interpret the system (2.1) and (2.2) as a sector of a more general system:

$$\tilde{S} = \tfrac{1}{2}\lambda^{-2}\int dt[(\dot{\rho})^2 - c^2(t)\rho^{-2}]$$

where we have introduced a non-propagating field $c(t)$ subjected to the constraint

$$\dot{c}(t) = 0 \Rightarrow c(t) = \text{constant}. \tag{A1.4}$$

Putting this constant equal to m one arrives at the action (2.1). Alternatively, one may implement (A1.4) in (A1.3) with the help of a Lagrange multiplier φ:

$$\tilde{S} \to \tilde{S}' = \tfrac{1}{2}\lambda^{-2}\int dt[(\dot{\rho})^2 - c^2(t)\rho^{-2} + 2c(t)\dot{\varphi}]. \tag{A1.5}$$

Instead of varying $\varphi(t)$, one may vary $c(t)$ to get

$$c(t) = -\rho^2\dot{\varphi}. \tag{A1.6}$$

After substituting this solution into (A1.5), the free $d = 1$ complex field action (A1.1) is reproduced.

Appendix 2. Comparison with the Hamiltonian form of $N = 4$ SCM [3]

In the original paper [3] from the beginning the quantum case was treated. However, no uncertainties appear upon taking the classical limit.

The Hamiltonian given in [3] is as follows:

$$H = \frac{1}{2}\left(p^2 + \frac{f^2}{x^2} + 2f\psi^+_\alpha\psi_\beta\frac{2x_\alpha x_\beta - x^2\delta_{\alpha\beta}}{x^4}\right)$$
$$= \tfrac{1}{2}[p_z p_{\bar{z}} + f^2(z\bar{z})^{-1} + \tfrac{1}{4}f(\bar{z})^{-2}\bar{\chi}\bar{\chi} + \tfrac{1}{4}fz^{-2}\chi\chi] \tag{A2.1}$$

where we have defined

$$z = x_1 + \mathrm{i}x_2 \qquad \chi_1 = \sqrt{2}\,(\psi_1^+ + \mathrm{i}\psi_2^+) \qquad \chi_2 = \sqrt{2}\,(\psi_1 + \mathrm{i}\psi_2).$$

Using the definition

$$\mathrm{i}\dot{A} = [A, H]$$

and canonical (anti)commutation relations, one finds the equations of motion to be

$$\begin{aligned} \ddot{z}(t) &= f^2 z^{-1}(\bar{z})^{-2} - \tfrac{1}{2} f(\bar{z})^{-3}\varepsilon_{ab}\bar{\chi}^a\bar{\chi}^b \\ \dot{\chi}_a &= \mathrm{i}f(\bar{z})^{-2}\varepsilon_{ab}\bar{\chi}^b. \end{aligned} \tag{A2.2}$$

These equations coincide with those following from the action (5.9) after identifying

$$mf = m\bar{f} = f \qquad z = Y_0 \tag{A2.3}$$

(one may always make f real by an appropriate phase transformation of spinor fields).

References

[1] De Alfaro V, Fubini S and Furlan G 1974 *Nuovo Cimento* A **34** 569
[2] Akulov V and Pashnev A 1983 *Teor. Mat. Fiz.* **56** 344
[3] Fubini S and Rabinovici E 1984 *Nucl. Phys.* B **245** 17
[4] Witten E 1981 *Nucl. Phys.* B **188** 513
[5] Smilga A 1985 *Nucl. Phys.* B **249** 413
[6] Smilga A 1986 *Nucl. Phys.* B **266** 45; 1987 *Nucl. Phys.* B **291** 241
[7] Gendenshtein E and Krive I 1985 *Usp. Fis. Nauk.* **146** 553
[8] Ivanov E, Krivonos S and Leviant V 1988 *Preprint* Geometry of conformal mechanics, JINR E2-88-370, Dubna; 1989 *J. Phys. A: Math. Gen.* **22** 345
[9] Ivanov E and Krivonos S 1983 *Lett. Math. Phys.* **7** 523; 1984 *Lett. Math. Phys.* **8** 39; 1984 *J. Phys. A: Math. Gen.* **17** L671
Ivanov E, Krivonos S and Leviant V 1988 *Nucl. Phys.* B **304** 601
[10] Gilmor R 1974 *Lie Groups, Lie Algebras and Their Applications* (New York: Wiley)
[11] Ramond P 1985 *Physica* **15D** 25
[12] Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2239
Callan C, Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2247
Volkov D V 1973 *Sov. J. Part. Nucl.* **4** 3
[13] Akulov V and Pashnev A 1985 *Teor. Mat. Fiz.* **65** 84
[14] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 313
[15] Gates S J, Grisaru M T, Roček M and Siegel W 1983 *Superspace* (Reading, MA: Benjamin/Cummings)
[16] Pashnev A I 1986 *Teor. Mat. Fiz.* **69** 311
[17] Kowalski-Glikman J, van Holten J W, Aoyama S and Lukierski J 1988 *Phys. Lett.* **201B** 487